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Lorentz manifolds modelled on a Lorentz symmetric space

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Abstract. *We give examples of Lorentz manifolds modelled on an indecomposable Lorentz symmetric space which are geodesically complete and not locally homogeneous.*

1. INTRODUCTION

Let (M, g) be a smooth, connected, pseudo-Riemannian manifold of dimension m and signature (p, q), $p + q = m$, and let R be its Riemann curvature tensor. Further, let (M_0, g_0) be a pseudo-Riemannian symmetric space of the same dimension m and the same signature (p, q) and denote by R_0 its Riemann curvature tensor. We say that (M, g) is *modelled* on (M_0, g_0) if for any point $\xi \in M$, there exists an isometry

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$\varphi_\xi : T_\xi M \to T_{u_0} M_0$) (u_0 being a base point of M_0) such that

$$\varphi_\xi^* R_{0,u_0} = R_\xi .$$

In particular this implies that (M, g) is *curvature homogeneous* in the sense of [10] (see also [6]). It has been proved in [11] that when (M_0, g_0) is an irreducible Riemannian symmetric space ($(p,q) = (m,0)$ or $(0,m)$), then (M,g) is necessarily locally symmetric and hence locally isometric to the model space (M_0, g_0).

The aim of this note is to investigate to what extent this situation generalizes to Lorentz type manifolds. In pseudo-Riemannian manifolds the notion of *irreducibility* has to be replaced by the weaker notion of *indecomposability* (or equivalently, of *weakly irreducibility*). Recall that (M,g) is said to be indecomposable if the holonomy group χ_ξ at the point ξ stabilizes only non-trivial subspaces V of $T_\xi M$ such that $g_{\xi|V\times V}$ is degenerate (i.e. has rank strictly smaller than dim V). The de Rham-Wu theorem [12] then asserts that any simply connected, complete, pseudo-Riemannian manifold is isometric to a product of indecomposable ones.

We will show that when (M,g) is modelled on an indecomposable, not (isotropy) irreducible Lorentz symmetric space (M_0, g_0), (M, g) is in general not locally symmetric. In fact, we construct for all dimensions $m \geq 3$ such spaces (M,g) which are not even locally homogeneus. The isometry class of such (M,g) depends on arbitrary functions. But up to now we are unable to determine, even locally, all manifolds (M, g) modelled on a given indecomposable, not irreducible Lorentz symmetric space.

Physically relevant models of classical general relativity should be examined from this point of view: a first question would be to determine to what extent the algebraic invariants of the curvature tensor determine locally the metric. There are examples of four-dimensional Lorentz manifolds having the same invariants without being locally isometric; but what is, at least to us, not known is the degree of arbitrariness of the metrics having prescribed invariants.

2. FIRST RESULTS

Indecomposable Lorentz symmetric spaces are classified by the following two theorems.

THEOREM 1. [2]. *Let (M_0, g_0) be a simply connected, indecomposable Lorentz symmetric space of dimension $m = n+2$, $n \geq 1$. Then either (M_0, g_0) is irreducible and admits a semi-simple isometry group or (M_0, g_0) admits a solvable transvection group. Furthermore, in this last case, $M_0 = \mathbf{R}^{n+2}$ and there exist n real numbers $\lambda_j, j \leq n$, such that*

(i) $\lambda_1 \cdots \lambda_n \neq 0$,

(ii) $\lambda_1^2 + \ldots + \lambda_n^2 = 1$
and the metric g_0 *is given by*

$$(2.1)\qquad g_0 = \left(\sum_{i=1}^{n} \lambda_i (x^i)^2\right) dx^0 \otimes dx^0 + dx^0 \otimes dx^{\underline{0}} + dx^{\underline{0}} \otimes dx^0 - \\ - \sum_{i=1}^{n} dx^i \otimes dx^i$$

where $x^0, x^{\underline{0}}, x^i, i \leq n$, *are the coordinates of* $\mathbf{R}^{n+2}$. ∎

THEOREM 2. *Let* (M_0, g_0) *be an irreducible Lorentz symmetric space of dimension* $m = n+2, n \geq 1$. *Then* (M_0, g_0) *has constant sectional curvature.*

∎

Proof. We use Berger's list of irreducible pseudo-Riemannian symmetric spaces [1] and compute in each case the signature. The list of these signatures is given in the Appendix. The result then follows at once from it.

REMARK. It would certainly be worthwile to have a direct proof of Theorem 2.

A direct calculation shows that the only non-vanishing components of the curvature tensor of (2.1) are

$$(2.2)\qquad R_{0j0j} = -\lambda_j, \quad 1 \leq j \leq n.$$

This implies that the Ricci tensor r has only one non-zero component

$$(2.3)\qquad r_{00} = \sum_{k=1}^{n} \lambda_k$$

and the scalar curvature

$$(2.4)\qquad \rho = 0 .$$

Theorem 1 and the above formulas imply the following

PROPOSITION 1. *Let* (M, g) *be a connected Lorentz manifold of dimension* $m = n+2, n \geq 1$, *modelled on the indecomposable, non-irreducible Lorentz symmetric space* (M_0, g_0) . *Then for all* $x \in M$, *there exists an «orthonormal» frame* $\{e_0, e_{\underline{0}}, e_j, 1 \leq j \leq n\}$ *where the only non-zero scalar products are*

$$g_{0\underline{0}} = 1, \quad g(e_j, e_j) = -1, \quad 1 \leq j \leq n$$

and such that with respect to the dual one-forms $\omega^0, \omega^{\underline{0}}, \omega^j$ of these basic vectors the curvature tensor is given by

$$R = -\sum_{j=1}^{n} \lambda_j (\omega^0 \wedge \omega^j) \otimes (\omega^0 \wedge \omega^j) \; . \tag{2.2'}$$

Moreover, the Ricci tensor has the form

$$r = \sum_{j=1}^{n} \lambda_j \omega^0 \otimes \omega^0 \tag{2.3'}$$

and the scalar curvature vanishes. ∎

REMARK. The line $\mathbf{R}e_{\underline{0}}$ is uniquely characterized by the property $R(X,Y)e_{\underline{0}} = 0$ for all vectors X, Y. (2.3') shows that the vector field $e_{\underline{0}}$ is globally defined up to sign. Further, from the contracted Bianchi identities one sees that

$$\nabla_{e_{\underline{0}}} e_{\underline{0}} = -(\operatorname{div} e_{\underline{0}}) e_{\underline{0}} \; . \tag{2.5}$$

Hence the integral curves of $e_{\underline{0}}$ are geodesics.

On the other hand, Theorem 2 implies

PROPOSITION 2. *A connected Lorentz manifold (M, g) modelled on an irreducible symmetric space (M_0, g_0) has constant sectional curvature.* ∎

3. EXPLICIT EXAMPLES

Let us consider a three-dimensional Lorentz manifold (M, g) modelled on a symmetric, indecomposable, non-irreducible space (M_0, g_0). Its curvature tensor is given by (2.2'). We shall assume that the characteristic vector field $e_{\underline{0}}$ is recurrent, that is

$$\nabla e_{\underline{0}} = \alpha \otimes e_{\underline{0}} \tag{3.1}$$

for a certain one-form α. From the contracted Bianchi identities one then deduces that $\alpha(e_{\underline{0}}) = 0$ and hence (2.5) becomes

$$\nabla_{e_{\underline{0}}} e_{\underline{0}} = 0 \; . \tag{3.2}$$

From (3.1) and the form of the Ricci tensor one deduces that (M, g) has recurrent curvature

$$\nabla R = 2\,\alpha \otimes R \; . \tag{3.3}$$

It then follows from [9, p. 169]:

PROPOSITION 3. *Each point x of (M,g) admits a neighborhood U_x in which there exists coordinates $x^0, x^{\underline{0}}, x^1$ such that the metric g is given by*

$$g = dx^0 \otimes (dx^{\underline{0}} + A dx^0) + (dx^{\underline{0}} + A dx^0) \otimes dx^0 - dx^1 \otimes dx^1 \tag{3.4}$$

where A is a function of two variables,

$$A = A(x^0, x^1) ,$$

such that $\partial^2 A/(\partial x^1)^2 \neq 0$ in all points of U_x. Furthermore, (M,g) is locally symmetric if and only if $\partial^2 A/(\partial x^1)^2$ is a constant. ■

REMARK. If we put

$$\omega^0 = h dx^0 ,\ \omega^{\underline{0}} = \frac{1}{h}(dx^{\underline{0}} + A dx^0),\ \omega^1 = dx^1 \tag{3.5}$$

where

$$h^2 = \frac{1}{\lambda_1} \frac{\partial^2 A}{(\partial x^1)^2},\quad \lambda_1 = \text{constant} , \tag{3.6}$$

then we have

$$g = \omega^0 \otimes \omega^{\underline{0}} + \omega^{\underline{0}} \otimes \omega^0 - \omega^1 \otimes \omega^1 \tag{3.7}$$

and

$$R = -\lambda_1 (\omega^0 \wedge \omega^1) \otimes (\omega^0 \wedge \omega^1) , \tag{3.8}$$

$$\alpha = d \ln h . \tag{3.9}$$

To construct a complete manifold (M,g) modelled on (M_0, g_0) we shall assume (i) that M is diffeomorphic to $\mathbf{R}^3$; (ii) that the metric (3.4) is a global one and (iii) that $A = A(x^1)$ with $\partial^2 A(\partial x^1)^2 > 0$. The equations of the geodesics of (M,g) are the Euler-Lagrange equations corresponding to the Lagrange functions

$$L = 2\frac{dx^0}{dt}\left(\frac{dx^{\underline{0}}}{dt} + A\frac{dx^0}{dt}\right) - \left(\frac{dx^1}{dt}\right)^2 .$$

These equations admit three first integrals:

$$\text{(a)}\quad 2\frac{dx^0}{dt}\left(\frac{dx^{\underline{0}}}{dt} + A\frac{dx^0}{dt}\right) - \left(\frac{dx^1}{dt}\right)^2 = \varepsilon ,$$

$$\text{(b)}\quad \frac{dx^0}{dt} = a,\ \frac{dx^{\underline{0}}}{dt} = b - 2Aa ,$$

where a, b are constants determined by the initial conditions and $\varepsilon = +1, -1, 0$ if the geodesic is time-like, space-like or null. Substituting (b) in (a) and integrating one gets

$$\text{(c)} \quad t = \int_{x^1_{(0)}}^{x^1} \frac{\pm dx^1}{\sqrt{(2ab - \varepsilon) - 2a^2 A}} .$$

The manifold (M, g) is complete provided any geodesic going to infinity, does it in an infinite time t. This means that the integral (c) must diverge when x^1 tends to $\pm\infty$. In particular this is the case if A has an asymptotic behavior of the form $-\frac{c^2}{2a^2}|x^1|^\lambda$ where c is a constant and $0 \leq \lambda \leq 2$.

We shall say that a function $A(x^1)$ is *admissible* if

(i) $\frac{\partial^3 A}{(\partial x^1)^3} \neq 0$ except in a finite number of points;

(ii) $\frac{\partial^2 A}{(\partial x^1)^2} > 0$ everywhere;

(iii) for $|x^1|$ large, A behaves like $-\frac{c^2}{2a^2}|x^1|^\lambda$, $0 \leq \lambda \leq 2$.

Then we have

PROPOSITION 4. *The manifold $(\mathbf{R}^3, ds^2)$ where*

$$\text{(3.10)} \qquad ds^2 = dx^0 \otimes (dx^{\underline{0}} + A dx^0) + (dx^{\underline{0}} + A dx^0) \otimes dx^0 - dx^1 \otimes dx^1 ,$$

and where A is an admissible function of x^1, is a complete Lorentz manifold modelled on (M_0, g_0).

■

Next, we prove

PROPOSITION 5. *Let $(\mathbf{R}^3, g)$ be a manifold where g is of type (3.4) with $A = A(x^1)$. Then it is locally homogeneous if and only if*

(i) *it is locally symmetric, or*

(ii) $A = a e^{b x^1}$, *where a and b are non-vanishing constants. In this last case $(\mathbf{R}^3, g)$ is globally homogeneous.*

Proof. We first note that for (3.5) we get

$$\text{(3.11)} \qquad \begin{cases} \nabla \omega^0 = \alpha \otimes \omega^0 , \\ \nabla \omega^{\underline{0}} = -\alpha \otimes \omega^{\underline{0}} - \beta \otimes \omega^1 , \\ \nabla \omega^1 = -\beta \otimes \omega^0 , \end{cases}$$

where

$$\text{(3.12)} \qquad \beta = h^{-2} \frac{\partial A}{\partial x^1} \omega^0 .$$

Next, suppose $(\mathbf{R}^3, g)$ is locally homogeneous and let f be a local isometry. Since $\mathbf{R}$ and ∇R are invariant under local isometries, we get from (3.3) that α is also invariant. This implies

$$||\alpha||^2 = -c^2$$

where

$$c = \frac{\partial \ln h}{\partial x^1}, \; c \text{ a constant} \tag{3.13}$$

because of (3.6), (3.9) and the assumption that $A = A(x^1)$.

If $c = 0$, then $\alpha = 0$ and $(\mathbf{R}^3, g)$ is locally symmetric.

Further, let $c \neq 0$. Then the invariance of α implies the invariance of dx^1 and hence

$$f^* x^1 = x^1 + m, \; m \text{ constant} . \tag{3.14}$$

Then, taking once more into account the invariance of R, we get

$$f^* \omega^0 = \epsilon \omega^0, \quad \epsilon^2 = 1 , \tag{3.15}$$

and then $f^* g = g$ implies

$$f^* \omega^{\underline{0}} = \epsilon \omega^{\underline{0}} . \tag{3.16}$$

Next, since $\nabla^2 R$ is also invariant, we obtain the invariance of $\nabla \alpha$, or equivalently, of $\nabla \omega^1$. Then (3.11), (3.12) and (3.15) yield

$$f^* \left(h^{-2} \frac{\partial A}{\partial x^1} \right) = h^{-2} \frac{\partial A}{\partial x^1} .$$

This and the local homogeneity condition yield that $h^{-2} \frac{\partial A}{\partial x^1}$ is constant and hence, using (3.6) we get

$$A = a e^{b x^1}, \quad ab \neq 0, \quad b = 2c . \tag{3.17}$$

Fom this, (3.14), (3.15) and (3.16) we get

$$\begin{cases} f^* x^1 = x^1 + m, \\ f^* x^0 = \epsilon e^{-cm} x^0 + p, \\ f^* x^{\underline{0}} = \epsilon e^{cm} x^{\underline{0}} + q, \end{cases} \tag{3.18}$$

where m, p, q are constants. This proves the required result.

Finally, we note that a diffeomorphism $f : \mathbf{R}^3 \to \mathbf{R}^3$ is an isometry between $(\mathbf{R}^3, ds^2)$ and $(\mathbf{R}^3, d\tilde{s}^2)$ where

$$d\tilde{s}^2 = 2\,dy^0(dy^{\underline{0}} + \tilde{A}(y^1)dy^0) - (dy^1)^2$$

if and only if

$$f(x^0, x^{\underline{0}}, x^1) = (ax^0 + c^0, \frac{1}{a}x^{\underline{0}} + bx^0 + c^{\underline{0}}, \epsilon x^1 + c^1)\,, \tag{3.19}$$

$$(\tilde{A} \circ f)(x^0, x^{\underline{0}}, x^1 = \tilde{A}(\epsilon x^1 + c^1) = \frac{1}{a^2}A(x^1) - \frac{1}{a}b\,, \tag{3.20}$$

where $\epsilon^2 = 1$ and $a \neq 0, b, c^0, c^{\underline{0}}, c^1$ are constants. Therefore, the isometry classes of the metric (3.10) are parametrized by the orbits of the functions A under the action of a Lie group G of dimension 5 which is the semi-direct product of the two-dimensional solvable group with the translation group $\mathbf{R}^3$. The method to prove this is similar to the one used in the proof of Proposition 5 (see also [6]). ■

4. REMARKS

(i) In dimension four one sees in virtue of Poposition 1 that any manifold (M, g) modelled on an indecomposable, non-irreducible symmetric space (M_0, g_0) has a conformal curvature tensor of Petrov type $\mathcal{N}$ [8] and a Ricci tensor corresponding to the energy distribution of a null electromagnetic field whose propagation vector coincides with the null characteristic vector of the conformal tensor. Indeed one deduces from (2.2') and (2.3') that there exists at each point a vector (here denoted $e_{\underline{0}}$) such that $i(e_{\underline{0}})C = 0$ (where C denotes the conformal curvature tensor); this is precisely the characterization of Petrov type $\mathcal{N}$. Conversely, any four-dimensional manifold wich is Ricci flat, of Petrov type $\mathcal{N}$ and is such that the curvature tensor is nonwhere zero, is modelled on a symmetric space; this is a direct consequence of the fact that there exists a change of adapted frame (Lorentz transformation) at each point which reduces the non-vanishing components of the curvature tensor (R_{0101} and R_{0202}) to opposite constants $(\lambda, -\lambda)$. Many space times of this form are known [3], [4], [7]; complete non-homogeneous examples should be characterized.

(ii) Proposition 3 and 4 generalize to arbitrary dimension $m \geq 4$ choosing $M = \mathbf{R}^m$ and as metric

$$ds^2 = 2\,dx^0(dx^{\underline{0}} + Adx^0) - \sum_{i=1}^{m-2}(dx^i)^2$$

where

$$A = \sum_{i=1}^{m-2} A_i(x^i)$$

and every A_i is an admissible function.

5. APPENDIX: PROOF OF THEOREM 2

One finds in [1] the list of all pairs $(\mathcal{G}, \mathcal{K})$ associated with those symmetric spaces G/K which have a simple isometry group. It is known that:

– if $\mathcal{K}$ contains a factor $\mathbf{R} = so(1,1)$, the space G/K is reducible and the invariant bilinear forms on it are of signature (p,p) ;

– if $\mathcal{K}$ contains a factor $T = so(2)$, the space G/K is pseudo-Kählerian and the invariant forms on it are of signature $(2p, 2q)$;

– if $\mathcal{G}$ and $\mathcal{K}$ are both pseudo-complex, the space G/K is $\mathbb{C}$-symmetric and it admits a two-parameter family of invariant forms the signature of which is (p,p) ;

– if $\mathcal{G} = \mathcal{K} \otimes \mathbb{C}$, the invariant forms on G/K have the signature of the Killing form of $\mathcal{K}$, which is Lorentzian only in the case $SL(2,\mathbb{C})/SL(2,\mathbf{R})$.

The signature of the spaces given in [1] and which do not belong to any of these classes is indicated in the following table (the notation for the real simple Lie algebras is defined in [5, p. 518]).

$\mathcal{G}$	$\mathcal{K}$	signature
$sl(n,\mathbf{R})$	$so(i,n-i)$	$(\frac{n}{2}(n+1) - i(n-i) - 1,$ $i(n-i))$
$sl(2n,\mathbf{R})$	$sp(n,\mathbf{R})$	(n^2-1, n^2-n)
$su^*(2n)$	$sp(i,n-i)$	$(2n^2 - n - 1 - 4i(n-i),$ $4i(n-i))$
$su^*(2n)$	$so^*(2n)$	(n^2-1, n^2+n)
$su(i,n-i)$	$so(i,n-i)$	$(i(n-i),$ $\frac{n}{2}(n+1) - 1 - i(n-i))$
$su(n,n)$	$so^*(2n)$	(n^2+n, n^2-1)
$su(n,n)$	$sp(n,\mathbf{R})$	(n^2-n, n^2-1)
$so^*(2n)$	$so^*(2i) + so^*(2n-2i)$	$(2i(n-i), 2i(n-i))$
$so^*(2n)$	$so(n,\mathbb{C})$	$(\frac{n}{2}(n-1), \frac{n}{2}(n+1))$
$so(i,n-i)$	$so(k,h) + so(i-k, n-i-h)$	$(k(n-h-i) + h(i-k),$ $k(i-k) + h(n-h-i))$
$so(m,m)$	$so(m,\mathbb{C})$	$(\frac{m}{2}(m+1), \frac{m}{2}(m-1))$
$sp(n,\mathbf{R})$	$sp(i,\mathbf{R}) + sp(n-i,\mathbf{R})$	$(2i(n-i), 2i(n-i))$
$sp(2n,\mathbf{R})$	$sp(n,\mathbb{C})$	$(n(2n+1), n(2n-1))$

$sp(i, n-i)$	$sp(k,h) + sp(i-k, n-i-h)$	$(4k(n-h-i)$ $+4h(i-k), 4k(i-k)$ $+4h(n-h-i))$
$sp(n,n)$	$sp(n,\mathbb{C})$	$(n(2n-1), n(2n+1))$
$G_{2(2)}$	$sl(2,\mathbb{R}) + sl(2,\mathbb{R})$	$(4,4)$
$F_{4(4)}$	$so(4,5)$	$(8,8)$
$F_{4(4)}$	$sp(1,2) + su(2)$	$(20,8)$
$F_{4(4)}$	$sp(3,\mathbb{R}) + sl(2,\mathbb{R})$	$(14,14)$
$F_{4(-20)}$	$so(1,8)$	$(8,8)$
$F_{4(-20)}$	$sp(1,2) + su(2)$	$(8,20)$
$E_{6(6)}$	$F_{4(4)}$	$(14,12)$
$E_{6(6)}$	$su^*(6) + su(2)$	$(28,12)$
$E_{6(6)}$	$sp(2,2)$	$(26,16)$
$E_{6(6)}$	$sp(4,\mathbb{R})$	$(22,20)$
$E_{6(6)}$	$sl(6,\mathbb{R}) + sl(2,\mathbb{R})$	$(20,20)$
$E_{6(2)}$	$su(2,4) + su(2)$	$(24,16)$
$E_{6(2)}$	$su(3,3) + sl(2,\mathbb{R})$	$(20,20)$
$E_{6(2)}$	$sp(1,3)$	$(28,14)$
$E_{6(2)}$	$F_{4(4)}$	$(12,14)$
$E_{6(2)}$	$sp(4,\mathbb{R})$	$(20,22)$
$E_{6(-14)}$	$F_{4(-20)}$	$(16,10)$
$E_{6(-14)}$	$su(2,4) + su(2)$	$(16,24)$
$E_{6(-14)}$	$sp(2,2)$	$(16,26)$
$E_{6(-14)}$	$su(4,2) + sl(2,\mathbb{R})$	$(20,20)$
$E_{6(-26)}$	$sp(1,3)$	$(14,18)$
$E_{6(-26)}$	$su^*(6) + su(2)$	$(12,28)$
$E_{6(-26)}$	$F_{4(-20)}$	$(10,16)$
$E_{7(7)}$	$so^*(12) + su(2)$	$(40,24)$
$E_{7(7)}$	$so(6,6) + sl(2,\mathbb{R})$	$(32,32)$
$E_{7(7)}$	$su(4,4)$	$(38,32)$
$E_{7(7)}$	$sl(8,\mathbb{R})$	$(35,35)$
$E_{7(7)}$	$su^*(8)$	$(43,27)$
$E_{7(-5)}$	$so(4,8) + su(2)$	$(32,32)$
$E_{7(-5)}$	$su(4,4)$	$(32,38)$
$E_{7(-5)}$	$su(2,6)$	$(40,30)$
$E_{7(-5)}$	$so^*(12) + sl(2,\mathbb{R})$	$(32,32)$
$E_{7(-25)}$	$su^*(8)$	$(27,43)$
$E_{7(-25)}$	$so(2,10) + sl(2,\mathbb{R})$	$(32,32)$
$E_{7(-25)}$	$su(2,6)$	$(30,40)$
$E_{7(-25)}$	$so^*(12) + su(2)$	$(24,40)$

$E_{8(8)}$	$E_{7(-5)} + su$	$(64, 48)$
$E_{8(8)}$	$so(8, 8)$	$(64, 64)$
$E_{8(8)}$	$E_{7(7)} + sl(2, \mathbf{R})$	$(56, 56)$
$E_{8(8)}$	$so^*(16)$	$(72, 56)$
$E_{8(-24)}$	$so^*(16)$	$(56, 72)$
$E_{8(-24)}$	$so(4, 12)$	$(64, 64)$
$E_{8(-24)}$	$E_{7(-5)} + su(2)$	$(48, 64)$
$E_{8(-24)}$	$E_{7(-25)} + sl(2, \mathbf{R})$	$(56, 56)$

The only Lorentzian irreducible spaces which appear in the above list are $SO_0(1, n)/SO_0(1, n-1)$ and $SO_0(2, n)/SO_0(1, n)$, both of constant curvature. Two other spaces are Lorentzian but a direct product, namely

$$SO^*(4)/SO(2, \mathbf{C}) \cong SU(2)/SO(2) \times SL(2, \mathbf{R})/\mathbf{R}$$

$$SO_0(2, 2)/SO(2, \mathbf{C}) \cong SL(2, \mathbf{R})/SO(2) \times SL(2, \mathbf{R})/\mathbf{R}.$$

On the other hand, the only irreducible Lorentzian symmetric space with a non-simple isometry group is $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})/SL(2, \mathbf{R}) \cong SO_0(2, 2)/SO_0(1, 2)$.

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